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The Area of the Solid of Intersection of a Sphere and an Ellipsoid, a Parametric Approach

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Abstract

In a previous paper we considered the intersection of a sphere and an ellipsoid using rectangular coordinates. In this paper we use a different approach based on using parametric coordinates and the use of a graphics program GInMA in order to gain further insight into this problem. As in our previous paper, we determine the surface area of the respective portions of the ellipsoid and the sphere that are inside each other. We provide examples to illustrate the various possibilities that arise and we provide Maple worksheets that can be used to deal with the calculations that must be performed. The task of the present paper is the derivation of the equations that allow us to represent graphically the solid of intersection and to calculate its surface area accurately and efficiently. We choose a system of coordinates with the origin at the center of the sphere and its axis directed toward the center of a spherical piece of the solid of intersection. We examine a variable step-size integration method. For an accuracy of approximately 10^{-6}, 100 *to* 600 *calculation points are typically sufficient and a typical calculation time is less than a minute.*

1 Problem Statement

We consider the equation of an ellipsoid centered at the point $O(0,0,0)$ in matrix form

$$\vec{X}^T A \vec{X} - 1 = 0, \quad \text{where} \quad A = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix}, \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1}$$

When it proves convenient, we use the ellipsoid in its scalar form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. We use a sphere of radius r centered at a point $\vec{X}_0^T = (x_0, y_0, z_0) = (k, l, m)$. We assume the center of the sphere is inside the first quadrant, where $x_0 \geq 0, y_0 \geq 0, z_0 \geq 0$ and that it is outside the ellipsoid. The z axis is chosen so that the ellipsoid vertex $(0, 0, c)$ is the vertex closest to point $\vec{X}_0$. We construct the solid of intersection of the sphere and the ellipsoid, render it, and find the surface area of the solid of intersection.

2 Possible Values of the Sphere Radius

Let d be the distance from $\vec{X}_0$ to the ellipsoid and let D be the greatest distance between $\vec{X}_0$ and points on the ellipsoid. A nontrivial solution is possible for our problem only if $d < r < D$. To find d and D, we use Lagrange's method for finding extrema. Finding conditional extrema of the function $(\vec{X} - \vec{X}_0)^2$ under the condition $\vec{X}^T A \vec{X} - 1 = 0$ is equivalent to considering the critical points for the function $f(\vec{X},\lambda)=(\vec{X}-\vec{X}_0)^2+\lambda(\vec{X}^T A\vec{X}-1)$. The equation for finding λ is expressible in the form of

$$\frac{x_0^2}{(a+\frac{\lambda}{a})^2}+\frac{y_0^2}{(b+\frac{\lambda}{b})^2}+\frac{z_0^2}{(c+\frac{\lambda}{c})^2}=1. \tag{2}$$

This equation is of sixth degree in λ. Since the center of the sphere is outside the ellipsoid; the equation has at least two real solutions but no more than 6 solutions. In this paper we consider the cases of two and four real roots. The equation (2) and [3] allow us to find the feet of perpendiculars dropped from the center of the sphere to the ellipsoid

$$\vec{X}_{H_i}=\left(\frac{x_0}{1+\lambda_i a^{-2}},\frac{y_0}{1+\lambda_i b^{-2}},\frac{z_0}{1+\lambda_i c^{-2}}\right). \tag{3}$$

These feet yield the radii of the sphere $|\vec{X}_{H_i}-\vec{X}_0|$ at points of tangency. The radius increases when λ decreases. Therefore, corresponds to the largest and D corresponds to the smallest permissible values of λ. Intermediate values of λ determine points of self-intersection on the curve of intersection of the sphere and ellipsoid. We use the locus of points $\vec{X}$ on the surface of the ellipsoid such that the normals at these points lie on a plane containing the centers of the sphere and the ellipsoid, that is, on the plane $O\vec{X}\vec{X}_0$. The equation of such points is given by $(\vec{X}-\vec{X}_0)\vec{X}\vec{X}_0=0$. In accordance with ([1], section 3.5.10, equation 3.5-22 of page 82) we denote the coordinates of the ellipsoid by

$$\begin{cases} x=a\sin u\cos v, \\ y=b\sin u\sin v, \\ z=c\cos u. \end{cases}$$

For some values of λ, the intersection curve is self-intersecting which we discuss in detail in Section 6.6. In the coordinates of the ellipsoid we obtain the following curve which has two branches:

$$c(a^2-b^2)z_0\tan u=\frac{a(c^2-b^2)x_0}{\cos v}+\frac{b(a^2-c^2)y_0}{\sin v}. \tag{4}$$

3 Intersection curve of the surfaces

The main difficulty in the construction of the intersection curve is associated with changes in the number of roots of the equation determining the curve when the parameters of the problem are changed. We try to choose a coordinate system so that the solution is unique. We use the following approach. Let the zenith of the spherical coordinate system pass inside the intersection curve through the center of the spherical part. In other words, let the points on the sphere corresponding to $\theta=\pi$ and zenith $\theta=0$ be divided by the intersection curve. Then for each φ in the interval $\varphi\in[0,2\pi)$ there is exactly one value of $\theta(\varphi)\in(0,\pi)$. In this case it is easy to find points of the intersection curve on the sphere. The points are converted using internal coordinates of the ellipsoid. It is intuitively clear that the calculation accuracy of the spherical piece area increases if the axis passes near the center of the spherical piece. It is clear that for $r\approx d$ the axis must pass through point $\vec{X}_H(d)$, and for $r\approx D$ the axis must pass through the point $\vec{X}_H(D)$. For the axis of the coordinate system for the sphere, a unit vector is in the direction of $\vec{X}_0\vec{A}$, where $\vec{A}=\vec{X}_H(d)+(\vec{X}_H(D)-\vec{X}_H(d))\frac{r-d}{D-d}$. In complex cases such as the one discussed in Section 6.6

we choose a point A manually. For the basis of the coordinate system X' we choose the center of the sphere to be $\vec{X}_0$. The axis of the coordinate system $X'(x', y', z')$ contains vectors

$$\vec{V}_1=\vec{V}_3\times\vec{V}_2, \vec{V}_2, \vec{V}_3, \text{ where } \vec{V}_3=\frac{\vec{A}-\vec{X}_0}{|\vec{A}-\vec{X}_0|}, \quad \vec{V}_2=(\vec{V}_{3y}, -\vec{V}_{3x}, 0). \tag{5}$$

The corresponding transformed coordinate system has the form

$$\vec{X}=M\,\vec{X}'+\vec{X}_0, \text{ where } M=(\vec{V}_1, \vec{V}_2, \vec{V}_3). \tag{6}$$

The equation of the ellipsoid is found by substituting (6) into (1):

$$\vec{X}'^T M^T A M\,\vec{X}'+2\vec{X}_0^T A M\,\vec{X}'+\vec{X}_0^T A\vec{X}_0-1=0. \tag{7}$$

We write the ellipsoid in the scalar form

$$a_{11}(x'^2)+a_{22}(y'^2)+a_{33}(z'^2)+2(a_{12}x'y'+a_{13}x'z'+a_{23}y'z'+a_{10}x'+a_{20}y'+a_{30}z')+a_{00}r^2=1,$$

where $a_{ij}=U_i^T U_j, U_0^T=(\frac{x_0}{ar}, \frac{y_0}{br}, \frac{z_0}{cr}), U_i^T=(\frac{V_{ix}}{a}, \frac{V_{iy}}{b}, \frac{V_{iz}}{c})$.

In other words, $a_{11}=\frac{V_{1x}^2}{a^2}+\frac{V_{1y}^2}{b^2}+\frac{V_{1z}^2}{c^2}, a_{22}=\frac{V_{2x}^2}{a^2}+\frac{V_{2y}^2}{b^2}+\frac{V_{2z}^2}{c^2}, a_{33}=\frac{V_{3x}^2}{a^2}+\frac{V_{3y}^2}{b^2}+\frac{V_{3z}^2}{c^2},$

$$a_{12}=\frac{V_{1x}V_{2x}}{a^2}+\frac{V_{1y}V_{2y}}{b^2}+\frac{V_{1z}V_{2z}}{c^2}, a_{13}=\frac{V_{1x}V_{3x}}{a^2}+\frac{V_{1y}V_{3y}}{b^2}+\frac{V_{1z}V_{3z}}{c^2},$$

$$a_{23}=\frac{V_{2x}V_{3x}}{a^2}+\frac{V_{2y}V_{3y}}{b^2}+\frac{V_{2z}V_{3z}}{c^2}, a_{10}=\frac{x_0V_{1x}}{a^2r}+\frac{y_0V_{1y}}{b^2r}+\frac{z_0V_{1z}}{c^2r},$$

$$a_{20}=\frac{x_0V_{2x}}{a^2r}+\frac{y_0V_{2y}}{b^2r}+\frac{z_0V_{2z}}{c^2r}, a_{30}=\frac{x_0V_{3x}}{a^2r}+\frac{y_0V_{3y}}{b^2r}+\frac{z_0V_{3z}}{c^2r}, \text{ and } a_{00}=\frac{x_0^2}{a^2r^2}+\frac{y_0^2}{b^2r^2}+\frac{z_0^2}{c^2r^2}. \tag{8}$$

Using the substitution ([1], section 3.1.6, equation 3.1-3 of page 60)

$$\begin{cases} x'=r\sin\theta\cos\varphi, \\ y'=r\sin\theta\sin\varphi, \\ z'=r\cos\theta, \end{cases} \tag{9}$$

we obtain the following equations relating θ and φ:

$$a_{11}\sin^2\theta\cos^2\varphi+a_{22}\sin^2\theta\sin^2\varphi+a_{33}\cos^2\theta+2(a_{12}\sin^2\theta\sin\varphi\cos\varphi+a_{13}\sin\theta\cos\theta\cos\varphi)+$$
$$+2(a_{23}\sin\theta\cos\theta\sin\varphi+a_{10}\sin\theta\cos\varphi+a_{20}\sin\theta\sin\varphi+a_{30}\cos\theta)+a_{00}=0,$$

$$\sin^2\theta(a_{11}\cos^2\varphi+a_{22}\sin^2\varphi+2a_{12}\sin\varphi\cos\varphi)+a_{33}\cos^2\theta+2\sin\theta\cos\theta(a_{13}\cos\varphi+a_{23}\sin\varphi)+$$
$$+2\sin\theta(a_{10}\cos\varphi+a_{20}\sin\varphi)+a_{30}\cos\theta+a_{00}=0, \tag{10}$$

Using the substitution

$$\begin{cases} \sin\theta=\frac{1-t^2}{1+t^2}\geq 0, \\ \cos\theta=\frac{2t}{1+t^2}, \end{cases} \tag{11}$$

we obtain the following equation

$$m_4t^4+(4a_{30}-m_2)t^3+2(2a_{33}-m_1+a_{00}-r^{-2})t^2+(4a_{30}+m_2)t+(m_4+2m_3)=0, \tag{12}$$

where $m_1 = a_{11}\cos^2\varphi + 2a_{12}\sin\varphi\cos\varphi + a_{22}\sin^2\varphi$, $m_2 = 4(a_{13}\cos\varphi + a_{23}\sin\varphi)$,

$m_3 = 2(a_{10}\cos\varphi + a_{20}\sin\varphi)$, $m_4 = m_1 - m_3 + a_{00} - r^{-2}$, and $\varphi \in [0, 2\pi)$.

The intersection curve on the sphere consists of points satisfying equation (12). Since any point of the intersection curve also belongs to the ellipsoid, we find its coordinates in the coordinate system X' by (12) and transform into the coordinate system X using (6). The internal coordinates of the ellipsoid (u, v) are then

$$\begin{cases} u = \arccos\frac{z}{c}, \\ v = \arccos\frac{x}{a\sin u}\cdot sign(y), v \in (-\pi, \pi]. \end{cases} \quad (13)$$

The equation of the intersection curve in the coordinates of the ellipsoid has the form

$$f(u, v) = (a\sin u\cos v - x_0)^2 + (b\sin u\sin v - y_0)^2 + (c\cos u - z_0)^2 - r^2 = 0. \quad (14)$$

If both distances from the sphere center $\vec{X}_0$ to the vertices $(0, 0, -c)$ and $(0, 0, c)$ are less than r or both distances are greater than r, there is a range of $v \in (v_{min}, v_{max})$, for which equation (14) has two real solutions. Equation (14) has one solution for $v = v_{min}$ and $v = v_{max}$. Outside this range, equation (14) does not have real solutions. In this case it is necessary to know the range of the variable v.

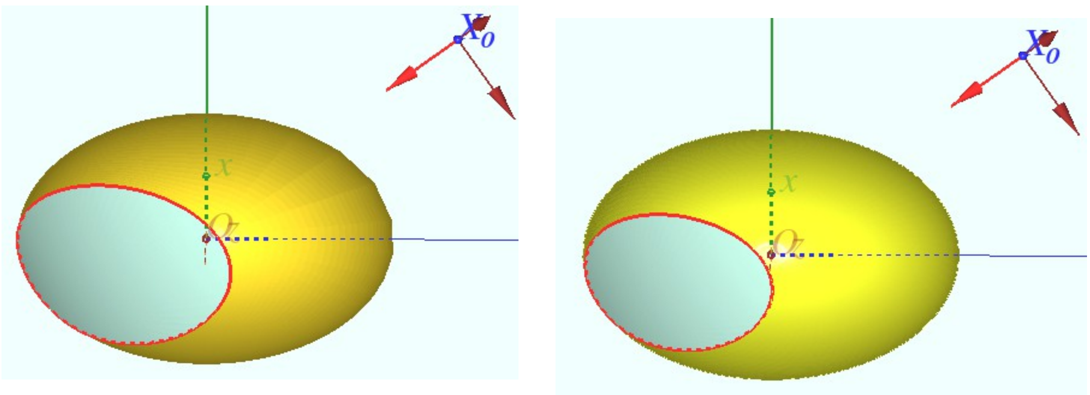

Figure 1: A view of the solid of the intersection along *z* axis for *r* = 8.1 (left) and *r* = 8.3 (right)

The Figure 1 shows the view of the solid of the intersection along *z* axis for *r* = 8.1 (left) and *r* = 8.3 (right), respectively. We see yellow ellipsoid, blue sphere, and red intersection curve. The axis of the coordinate system $X'(x', y', z')$ contains vectors $\vec{V}_1, \vec{V}_2$ (brown) and $\vec{V}_3$ (red) with the origin in the center of the sphere $\vec{X}_0$. The axes *x* (*y*) of the coordinate system of the ellipsoid is green (blue) and has origin in the center of the ellipsoid point *O*. The axes *z* projected in the point *O*. When the center of the sphere is at $\vec{X}_0^T = (3.2, 4.0, 2.4)$, the radius is 8.1, and the center of the ellipsoid is at $\vec{O}^T = (0,0,0)$ and semi-axes are (2,3,4), the intersection curve covers the z axis, and the center of the ellipsoid is being projected inside the blue area, which is part of the surface of the sphere. The parameter *v* accepts any value $v \in [0, 2\pi)$. This situation corresponds to Figures 8, 9, and 10 (middle figures) for *r* = 6, *r* = 7, and *r* = 8, respectively. In the case of *r* = 8.3, the intersection curve (red) does not cover the z axis, and the center of the ellipsoid is being projected outside the blue area. The parameter *v* shall vary in the range of $[v_{min}, v_{max}]$. This situation corresponds to the middle figures of Figures 4 to 7 and 11. The intersection curve is closed in the *u-v* plane. It is worth noting that at points when $v=v_{min}$ or $v=v_{max}$, the gradient vector of the function has a component only along the *v* axis. Consequently, we obtain the equation $\frac{\partial f(u,v)}{\partial u}=0$ which we rewrite using coordinates $\vec{X}$ as

$$x(x-x_0)+y(y-y_0)+(z-z_0)(z-\frac{c^2}{z})=0. \tag{15}$$

In practice it is convenient to solve this equation together with the equations of the sphere and ellipsoid and find (u_k, v_k) using $(x_k, y_k, z_k), k=1,2$ and the relations (13).

4 Basic Formulas

Let us consider the equations for the region on a quadric surface that is bounded by a given curve. Let S_1 be a smooth three dimensional surface of the form $f(u,v)=0$, where *u* and *v* range over some region Γ with *v* in the interval $[v_{min}, v_{max}]$ and *u* in the interval $[u_1(v), u_2(v)]$. The surface is defined parametrically by $(x(u,v), y(u,v), z(u,v))$.
A region being a portion of the surface corresponding to the mentioned region is

$$s=\iint_{(\Gamma)} J\, du\, dv,\ J=\sqrt{EG-F^2}\ \text{, where } E=(\frac{\partial x}{\partial u})^2+(\frac{\partial y}{\partial u})^2+(\frac{\partial z}{\partial u})^2,$$

$$G=(\frac{\partial x}{\partial v})^2+(\frac{\partial y}{\partial v})^2+(\frac{\partial z}{\partial v})^2, \qquad F=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}.$$

For all quadric surfaces this expression is integrated with respect to u.

Case 1

In case of a sphere, we have $f(x,y,z)=x^2+y^2+z^2-r^2=0$, $x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi$, $z=r\cos\theta$, $J=r^2\sin\theta$. We integrate this equation to obtain

$$s=\int_{\varphi_{min}}^{\varphi_{max}}\int_{\theta_1(\varphi)}^{\theta_2(\varphi)} r^2\sin\theta\, d\theta\, d\varphi=r^2\int_{\varphi_{min}}^{\varphi_{max}}(\cos\theta_2(\varphi)-\cos\theta_1(\varphi))\,d\varphi. \tag{16}$$

Case 2

In case of an ellipsoid, we have $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$,

$x=a\sin u\cos v, y=b\sin u\sin v, z=c\cos u$,
$J=\sin u\sqrt{a^2b^2\cos^2u+c^2\sin^2u(a^2\sin^2v+b^2\cos^2v)}$.

Let $t=\cos u$, $h(v)=a^2\sin^2v+b^2\cos^2v$, $H(v)=a^2b^2-c^2h(v)$, $h_1(v)=\sqrt{H(v)}$.

Then $J\,dt=\sqrt{t^2a^2b^2+c^2(1-t^2)h(v)}\,dt$.

Let $g(t,v)=t\sqrt{c^2h(v)+H(v)t^2}+\frac{c^2h(v)}{h_1(v)}\ln(h_1(v)t+\sqrt{c^2h(v)+H(v)t^2})$. Then we have

$$s=\int_{v_{min}}^{v_{max}}\int_{\sin u_1(v)}^{\sin u_2(v)}\sqrt{t^2a^2b^2+c^2(1-t^2)h(v)}\,dt\,dv=\ \frac{1}{2}\int_{v_{min}}^{v_{max}}(g(\sin u_2(v),v)-g(\sin u_1(v)),v)\,dv. \tag{17}$$

We use formulas (16) and (17) to find the surface areas of the intersection solids bounded by the curve.

5 Integration method: Examples

5.1. We consider the sphere of radius 2 whose center is at (0,0,5), and the ellipsoid has its semi-axis (2,3,4) and center is at (0,0,0) (see [6]). We note that $\theta=0$ corresponds to the *z*-axis for the sphere, and note that (θ,φ) is chosen such that θ defined by equations (11) and (12), is continuous. The curve of intersection of the sphere and ellipsoid is unique for any φ. The area of the portion of the sphere inside the ellipsoid is then

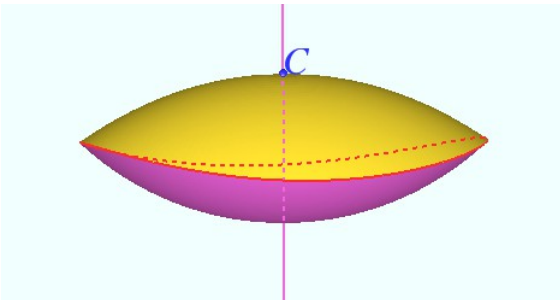

Figure 2: The solid of intersection, example 5.1

$$S_{sp}=\int_0^{2\pi}\int_0^{\theta(\varphi)} r^2\sin\theta\,d\theta\,d\varphi \;=\; r^2\int_0^{2\pi}(1-\cos\theta(\varphi))\,d\varphi. \tag{18}$$

The integration can be approximated using a constant increment $\varphi_i=\frac{2\pi i}{n}$. We find $\theta(\varphi_i)$ from (12). The integral sum according to Simpson's formula is

$$S(n)=\frac{4\pi r^2}{3n}\sum_{i=1}^{n} k_i(1-\cos\theta(\varphi_i)),\, k_{2i}=1, k_{2i+1}=2. \tag{19}$$

The accuracy of this approximation is proportional to n^{-5}. Therefore, a relative accuracy of 10^{-6} is obtained by using at 35-50 calculation points. The calculation time is less than 10 seconds.

The distance from the center of the sphere $\vec{X}_0$ to the vertex $(0,0,c)$ is smaller than the radius of the sphere. Therefore, the vertex $(0,0,c)$ and $u=0$ belong to the required surface. The distance from the center of the sphere $\vec{X}_0$ to the opposite vertex $(0,0,-c)$ is larger than the sphere radius. So the vertex $(0,0,-c)$ and $u=\pi$ is outside the surface. The surface area of the portion of the ellipsoid inside the sphere calculated by formula (17), where $\cos u_2(v)$ is 1. Then

$$S_{el}=\frac{1}{2}\int_0^{2\pi}(g(1,v)-g(\cos u(v),v))\,dv. \tag{20}$$

The total integral consists of the area of the curvilinear triangles for which Using Simpson's formula we see that

$$S(n)=\frac{2\pi}{3n}\sum_{i=1}^{n} k_i(g(0,v_i)-g(\cos u(v_i),v_i)), \tag{21}$$

where $v_i=\frac{2\pi i}{n}$, $k_i=1$ if *i* is even, and $k_i=2$ if *i* is odd.

The accuracy of the summation is again proportional to n^{-5}. If the number of calculation points is greater than 33, the error in the calculation is significantly less than 10^{-8}.

This example focuses on how to verify calculation accuracy since the integrals can be approximated in several ways. For example, in [2, Example 1], the area of the relevant portion of the sphere is found to be 5.403 and the area of the relevant portion of the ellipsoid is 5.821.

We note that the exact solution for the surface area of the sphere defined in this section has the 11 significant figure approximation to 5.4028467054, and we obtain an answer of 5.4028467060 by using the standard calculation from this section when 152 points are used. Similarly, the exact solution of the surface area for the ellipsoid has the 11 significant figure approximation to 5.8210351042, and we obtain an answer of 5.821035103 by standard calculation mentioned in this section when 152 points are used.

5.2. Assume the center of a sphere of radius r = 2.2573 is located at $\vec{X}_0=(1,2,3)$, and the ellipsoid semi axes are (2, 3, 4) (see [7]). The calculation is performed as described in section 5.1. The coordinate system for the sphere is set automatically by the equation (5). We find the surface area belonging to the ellipsoid to be approximately 13.8364, and the surface area belonging to the sphere to be approximately 13.83697. These values are in close agreement with those given in [2, Example 2], where it was reported that the surface areas for ellipsoid and sphere, when r = 2.2574, are approximately 13.827 and 13.838, respectively.

We assume the relative error in the calculation is based on the number of points n and the formula $\delta(n)=\frac{|S(n)-S|}{S}$, where S is the true value of the area. If the parameter $n^2\delta(n)$ is constant, the accuracy of the calculation is of order n^{-2}. Under this assumption, the estimation of the true value of the area S is obtained by the formula

$$S\approx S(2n)+\frac{S(2n)-S(n)}{3}. \tag{22}$$

For the area of the ellipsoid, the relative error in the present calculation $\delta(n)$ is approximately n^{-2}, and it should decrease with increasing n if the accuracy is higher. As shown in Table 1, the parameter $n^2\delta(n)$ is approximately constant.

n	52	152	302	602
Ellipsoid $\delta(n)n^2$	1.5	1.5	1.6	1.4
Sphere $\delta(n)n^2$ %	0.0005	<0.0001	<0.0001	<0.0001

Table 1

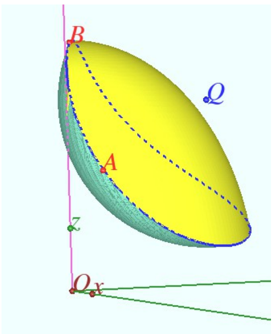

Figure 3: The solid of intersection, example 5.2

6 Integration issues

This section discusses important details of the integration methods used to determine the surface areas of the solid of intersection. These issues include the choice of the integration steps and increments, the manner in which certain extreme regions in the integration domains are treated, and the different summation methods used to obtain approximate values for the relevant integrals.

6.1 Solid of Intersection

Assume the center of a sphere is located at $\vec{X}_0=(3.2,4,2.4)$ and the ellipsoid semi axes are (2, 3, 4). The calculation of the area of the portion of the sphere inside the ellipsoid is performed as in the previous examples. For r = 2.9 the ellipsoid does not have a vertex which belongs to the solid of the intersection. So we need to use formula (17) for the calculations. The integral sum is determined by the fragments of the integration area. The fragmentation for r = 2.9 is shown schematically in the middle graph of Figure 4. It consists of a number of strips and the two end zones, shown in blue and red, respectively. The integral sum contains the areas of n curved strips $v\in[v_i, v_{i+1}], u\in[u_1(v_{i+1/2}), u_2(v_{i+1/2})], v_{i+1/2}=\frac{v_{i+1}+v_i}{2}$ and two segments of a circle, one of which is red and located between $v=v_{min}$ and v_1, and the other is located between $v=v_n$ and $v=v_{max}$, which is shown in blue.

6.2 Choice of the integration step

The typical shape of the integration domain somewhat resembles that of a circle. In order to place the calculated points fairly evenly on the curve we use the formula:

$$v_i=\frac{v_{max}+v_{min}}{2}+\frac{v_{max}-v_{min}}{2}\cos\frac{\pi i}{n+1}. \tag{23}$$

To check the accuracy with different choices of the integration model we carried out studies for the randomly chosen function $\zeta(x,y)=-\sin^2(x+2y)\cos(3x+2y)y$. We used some nontrivial functions and result was the same. We evaluated the integral

$$I=\int_{x^2+y^2\le 1}\frac{\partial\zeta(x,y)}{\partial y}dx\,dy.$$

The value of the integral calculated using Maple, is $I\approx 0.41992728$. For each x_i we calculate two values $\zeta_i=\zeta(x_i,-\sqrt{1-x_i^2})$, and $\zeta^\circ_i=\zeta(x_i,\sqrt{1-x_i^2})$. The difference between them is $\Delta\zeta_i=\zeta^\circ_i-\zeta_i$, $\Delta\zeta_1=\Delta\zeta_{n+1}=0$, using which we then find the integral sum. Table 2 shows the results of the calculations. Line 2 corresponds to the simple integral sum, $I_1(n)=\sum_1^n h\,\Delta\zeta_i$, where $h=\frac{2}{n}=x_{i+1}-x_i$. Line 3 corresponds to the Simpson integral sum $I_2(n)=\sum_1^n\frac{h}{3}k_i\Delta\zeta_i$, where $h=\frac{2}{n}$, $k_{2j}=2$, and $k_{2j+1}=4$. Line 4 corresponds to the integral sum $I_3(n)=\sum_1^{n-1}\frac{\Delta\zeta_{i+1}+\Delta\zeta_i}{2}(x_{i+1}-x_i)+\frac{2}{3}\Delta\zeta_1(x_1+1)+\frac{2}{3}\Delta\zeta_n(1-x_n)$. We performed numerical simulations using different numbers of points n and find the relative error in the calculation based on the number of points, that is, $\delta_k(n)=\frac{I-I_k(n)}{I}$, being expressed as a percentage. It is seen that our model calculation with a constant step leads to very large errors. The calculation of the selected variable step has significantly greater accuracy and typically has accuracy of order n^{-2}.

n	25	50	100	200
$\delta_1(n^2\delta_1)$	3.96% (2477)	1.5% (3729)	0.54% (5420)	0.19% (7761)
$\delta_2(n^2\delta_2)$	3.96% (2477)	2.31% (5788)	0.85% (8585)	0.31% (12400)
$\delta_3(n^2\delta_3)$	0.160% (100)	0.052% (130)	0.014% (147)	0.0039% (156)

Table 2

6.3 Estimation of the Area of the Extreme Portions of the Integration Domain

We turn our attention to an extreme portion of the integration domain located between $v=v_{min}$ and v_1. We approximate the intersection curve $f(u,v)=0$ using a quadratic function which passes through the points $(v_{min}, u(v_{min}))$, $(v_1, u_2(v_1))$, $(v_1, u_1(v_1))$. We use the following formula to approximate the area of the region of the parabola, which is used to approximate the surface area of the ellipsoid as follows:

$$ds_{el} \approx \frac{2}{3}(g(\cos(u_2(v_1)), v_1) - g(\cos(u_1(v_1)), v_1))(v_1 - v_{min}). \tag{24}$$

We estimate the area for the domain of integration between $v = v_n$ and $v = v_{max}$.

6.4 Integral sum formula

We consider a term of the integral sum $dS_i = \int_{x_i}^{x_{i+1}} y(x)dx$ for a variable integration step. We approximate the integrand in the interval $[x_i, x_{i+1}]$ using a quadratic function and require that the function interpolate the points (x_i, y_i) and (x_{i+1}, y_{i+1}) and that it is close to the points (x_{i-1}, y_{i-1}) and (x_{i+1}, y_{i+1}) in a least squares sense. Let $y_e(x)$ be estimation for $y(x)$:

$$y_e = \mu(x - x_i)^2 + (\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \mu(x_{i+1} + x_i))(x_{i+1} - x_i) + y_i, \tag{25}$$

where

$$\mu = -\frac{\xi(x_{i-1})\eta(x_{i-1}) + \xi(x_{i+2})\eta(x_{i+2})}{\xi^2(x_{i-1}) + \xi^2(x_{i+2})},$$

$$\xi(x) = x^2 + (x_{i+1} + x_i)x - x_{i+1}x_i,$$

$$\eta(x) = y_{i+1}\frac{x_i - x}{x_{i+1} - x_i} - y_i\frac{x_{i+1} - x}{x_{i+1} - x_i} - y(x).$$

$$dS_i \approx \int_{x_i}^{x_{i+1}} y_e(x)dx = \frac{y_{i+1} + y_i}{2}(x_{i+1} - x_i) - \frac{\mu}{6}(x_{i+1} - x_i)^3. \tag{26}$$

With the integral error of h^2 replaced by the integral sum trapezoidal method we obtain

$$dS_i = \int_{x_i}^{x_{i+1}} y(x)dx \approx \frac{y_{i+1} + y_i}{2}(x_{i+1} - x_i). \tag{27}$$

We estimate the sum of the form $S = dS(0) + \sum_{i=1}^{n} dS(i) + dS(n+1)$, where

$$dS(0) = \frac{2}{3}ds_1(v_1 - v_{min}), \quad dS(i) = \frac{ds_i + ds_{i+1}}{2}(v_{i+1} - v_i),$$

$$dS(n+1) = \frac{2}{3}ds_{n+1}(v_{max} - v_n), \quad v_i = \frac{v_{max} + v_{min}}{2} - \frac{v_{max} - v_{min}}{2}\cos\frac{i\pi}{n+1}, \text{ and obtain}$$

$$ds_i = g(\cos(u_{min}(v_i), v_i)) - g(\cos(u_{max}(v_i), v_i)). \tag{28}$$

In the calculations the coordinate system is defined by the equations of (4) and (5). Table 3 shows the values of the parameters $\delta(n) = 100(\frac{S(n)}{S} - 1), S \approx 2.209234$, and $n^2\delta(n)$ for $r = 3.1$.

n	25	50	100	200
S	2.203862	2.207886	2.208882	2.209146
$\delta(n)$	0.24%	0.061%	0.016%	0.004%
$\delta(n)n^2$	1.52	1.53	1.60	1.60

Table 3

6.5 Results

We perform calculations for the sphere with fixed center and a fixed ellipsoid. We change the radius of the sphere and the method of calculating the area of the ellipsoid (as well as the method of rendering its image in GInMA). In separate calculations, instead of the ellipsoid (a, b, c) and the sphere (k, l, m) we use the ellipsoid (a, c, b) and the sphere (k, m, l). We found the distance from the point X_0 to some of the vertices of the ellipsoid to be $d \approx 2.879132116$, $D \approx 8.770649392$. The closest vertex to the point $\vec{X}_0$ is at $(0, b, 0)$, a distance of $d_{min} \approx 4.123105626$. The distance to the vertex $(0, 0, c)$ is $d_c \approx 5.179880467$. The maximum distance to the vertex $(0, 0, -c)$ is 8.197560613.

The calculation results are made in [8] and shown in Figures 3-10 and Table 4. In the first column of Table 4, the specification of the ellipsoid and the radius of the sphere are given. $S(2, 3, 4)$ corresponds to $a=2, b=3, c=4$, and to the center of the sphere at the point $\vec{X}_0^T = (3.2, 4.0, 2.4)$. $S(2, 4, 3)$ corresponds to the change of the original coordinate system axes for $a=2, b=4, c=3$, and $\vec{X}_0^T = (3.2, 2.4, 4.0)$. Columns 2-4 correspond to different numbers of calculation points. Column 2 shows the calculation using 50 points, column 3 using 100 points, and column 4 using 200 points. Column 5 is obtained by extrapolation of formula (22). The top number in each cell gives the surface area of the ellipsoidal piece of the intersection solid while the lower number shows the surface area of the spherical piece. The pairs of calculations with the same radius for the sphere show how the accuracy of the calculation changes. For the radii $r=4.5$ and $r=5$, we have $d_{min} < r \le d_c$, and the intersection curve does not separate the vertices in the coordinate system for $a=2, b=3, c=4$. At the same time, in the coordinate system $S(2,4,3)$, the intersection curve divides the vertices. Therefore, the accuracy of calculations with a small number of calculation points varies considerably but the extrapolated results are the same.

The relevant figures are given after Table 4. Each figure contains three images. The left image obtained using Maple shows the curve of intersection using the coordinates of the sphere (θ, φ), with θ located on the horizontal axis and φ located on the vertical axis. The curve is of the same type in all of the figures; exactly one value $\theta(\varphi) \in (0, \pi)$ exists for any $\varphi \in [0, 2\pi)$.

The center image, obtained using Maple, shows the intersection curve in the coordinates of the ellipsoid (u, v), with u located on the horizontal axis and v located on the vertical axis. For small and large radii the curve is close to a circle. For intermediate radii $v \in [0, 2\pi)$ in order for the curve to be continuous, it must be the case that $v \in (-\pi, 2\pi)$, and values that differ by 2π correspond to the same point on the surface.

The right image, obtained using GInMA, shows the solid of the intersection. For small radii, for example $r=2.9$, the intersection solid looks like a disk and the surface area of the ellipsoid piece is slightly smaller than that of the spherical piece. By increasing the radius the difference between the surface areas decreases monotonically and when $r \approx 3.095$ the surface areas become the same. When the radius is approximately d_{min} the intersection solid is located close to the axis of ordinates. When $r = d_{min}$ the intersection curve passes through the point $(0, b, 0)$. Furthermore, the intersection solid monotonically becomes the ellipsoid.

Note that in the entry in Table 4 for $r=5$ and $S(2, 3, 4)$. In this case the border does not separate the ellipsoid vertices, the calculation accuracy is low and the correct result is obtained only by extrapolation or by using a number of points much greater than 300. On the other hand, when

$S(2,4,3)$, $\vec{X}_0^T=(3.2,2.4,4.0)$ and $r=4.5,5$ or 6, the boundary separates the vertices, the accuracy of computation will be achieved when n = 100. In other words, the respective portions of the surface areas for ellipsoid and sphere stay unchanged when $n \geq 100$ in this case.
On the other hand, we note for $r=7$ in both cases, the intersecting curve runs between the vertices of the ellipsoid, the accuracy for both cases stays the same.

n	50	100	200	∞
$S(2,3,4)$ $r=2.9$ $\vec{X}_0^T=(3.2,4.0,2.4)$	0.202102 0.20228133	0.202193 0.20228133	0.202217 0.20228133	0.202225 0.20228133
$S(2,3,4)$ $r=3.095$ $\vec{X}_0^T=(3.2,4.0,2.4)$	2.156223 2.15745038	2.157192 2.15745038	2.157446 2.15745038	2.157531 2.15745038
$S(2,3,4)$ $r=4$ $\vec{X}_0^T=(3.2,4.0,2.4)$	12.844892 12.040128	12.850634 12.040128	12.852143 12.040128	12.852643 12.040128
$S(2,3,4)$ $r=4.5$ $\vec{X}_0^T=(3.2,4.0,2.4)$	19.996242 17.327758	20.005151 17.327758	20.007496 17.327758	20.008278 17.327758
$S(2,4,3)$ $r=4.5$ $\vec{X}_0^T=(3.2,2.4,4.0)$.	20.008265 17.327758	20.008301 17.327758	20.008301 17.327758	20.008301 17.327758
$S(2,3,4)$ $r=5$ $\vec{X}_0^T=(3.2,4.0,2.4)$	28.124322 21.8750020	28.136841 21.87500120	28.140160 21.87500120	28.141237 21.87500120
$S(2,4,3)$ $r=5$ $\vec{X}_0^T=(3.2,2.4,4.0)$.	28.141305 21.875000	28.141270 21.875001	28.141270 21.875001	28.141270 21.875001
$S(2,3,4)$ $r=6$ $\vec{X}_0^T=(3.2,4.0,2.4)$	47.433904 27.004988	47.431412 27.005004	47.431404 27.005004	47.431401 27.005004
$S(2,4,3)$ $r=6$ $\vec{X}_0^T=(3.2,2.4,4.0)$.	47.431391 27.005004	47.431404 27.005004	47.431404 27.005004	47.431404 27.005004
$S(2,3,4)$ $r=7$ $\vec{X}_0^T=(3.2,4.0,2.4)$	70.265887 24.596703	70.265878 24.59663903	70.265878 24.59663903	70.265878 24.59663903
$S(2,4,3)$ $r=7$ $\vec{X}_0^T=(3.2,2.4,4.0)$.	70.265882 24.596704	70.265878 24.596639	70.265878 24.596639	70.265878 24.596639
$S(2,3,4)$ $r=8$ $\vec{X}_0^T=(3.2,4.0,2.4)$	94.5444568 13.562497	94.544456 13.56254318	94.544456 13.56254318	94.544456 13.56254318
$S(2,3,4)$ $r=8.5$ $\vec{X}_0^T=(3.2,4.0,2.4)$	105.925459 5.183001	105.922890 5.18299733	105.922218 5.18299733	105.922000 5.18299733

Table 4

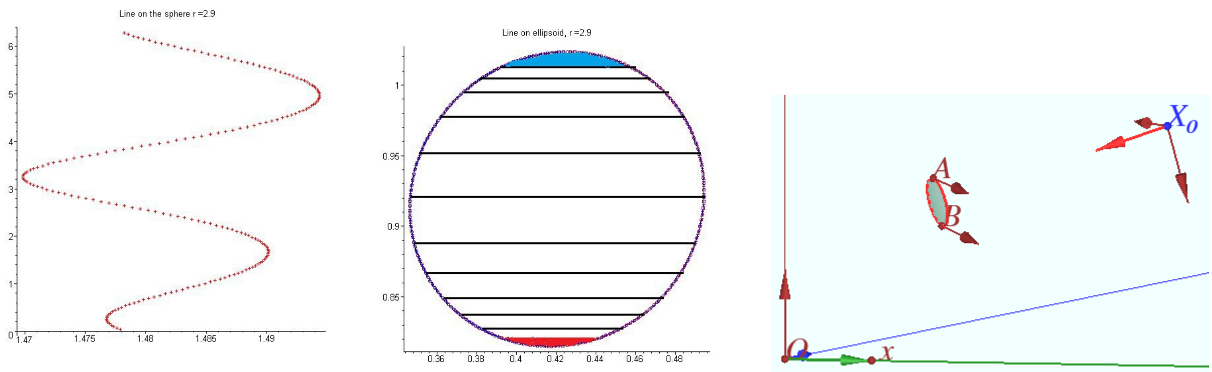

Figure 4: Intersection curves on the sphere and ellipsoid and intersection solid for $r = 2.9$

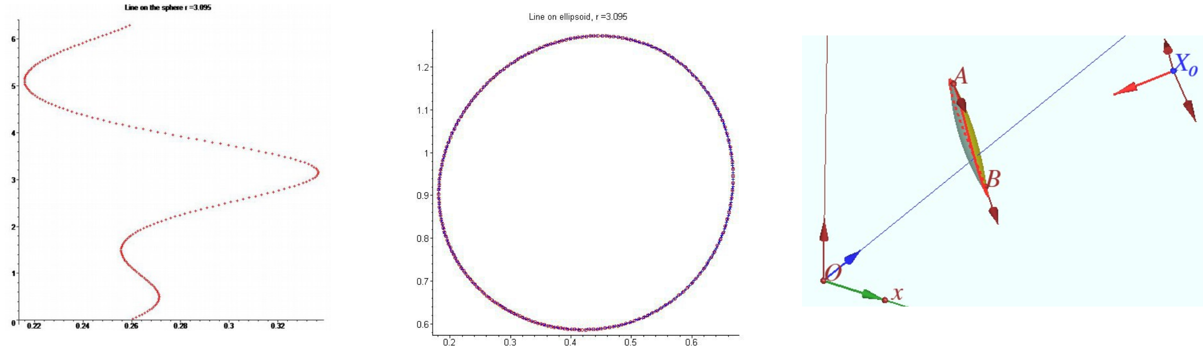

Figure 5: Intersection curves on the sphere and ellipsoid and intersection solid for $r = 3.095$

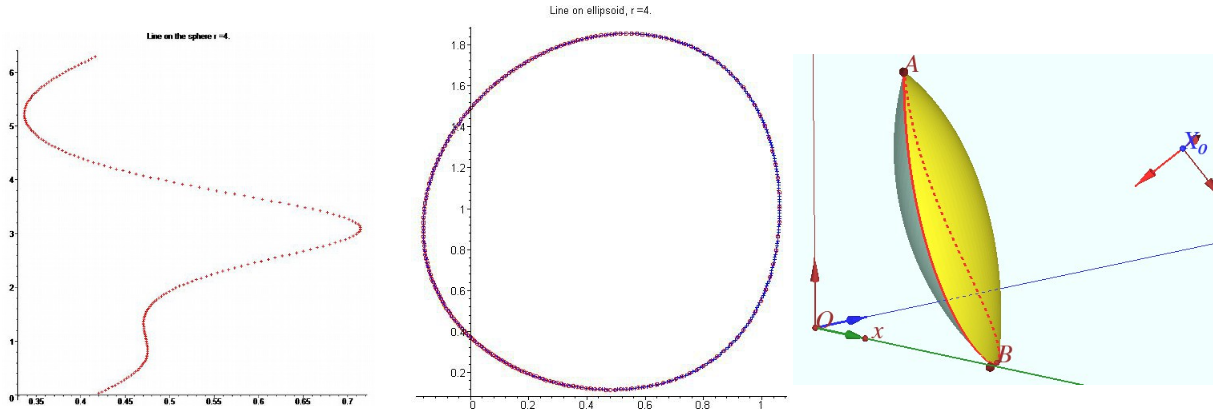

Figure 6: Intersection curves on the sphere and ellipsoid and intersection solid for $r = 4$

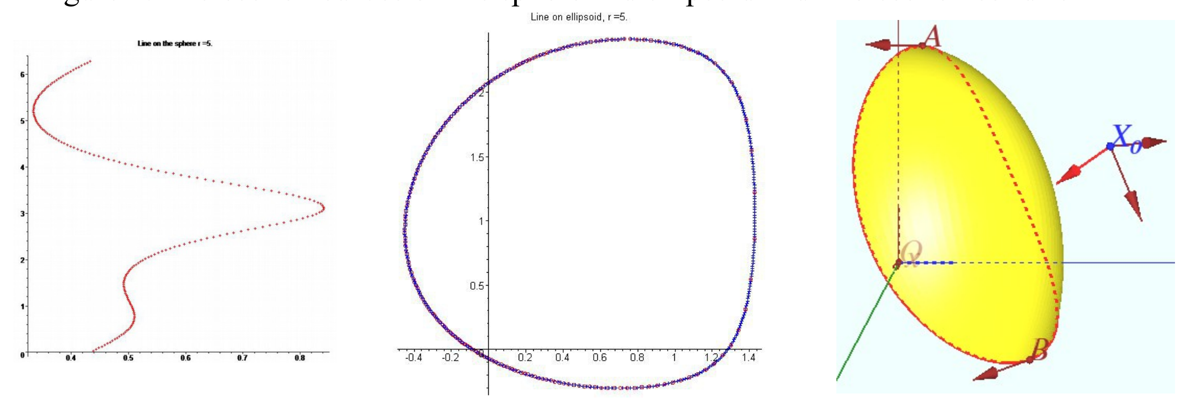

Figure 7: Intersection curves on the sphere and ellipsoid and intersection solid for $r = 5$

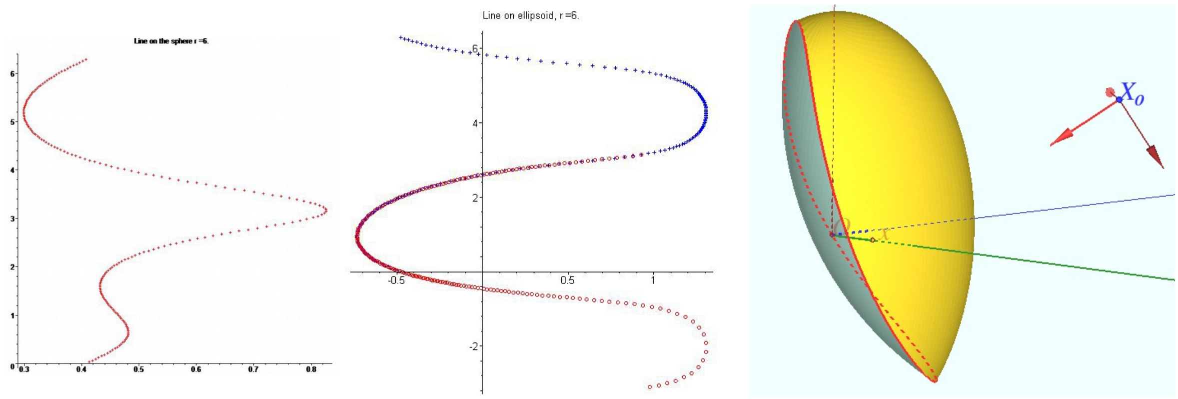

Figure 8: Intersection curves on the sphere and ellipsoid and intersection solid for $r=6$

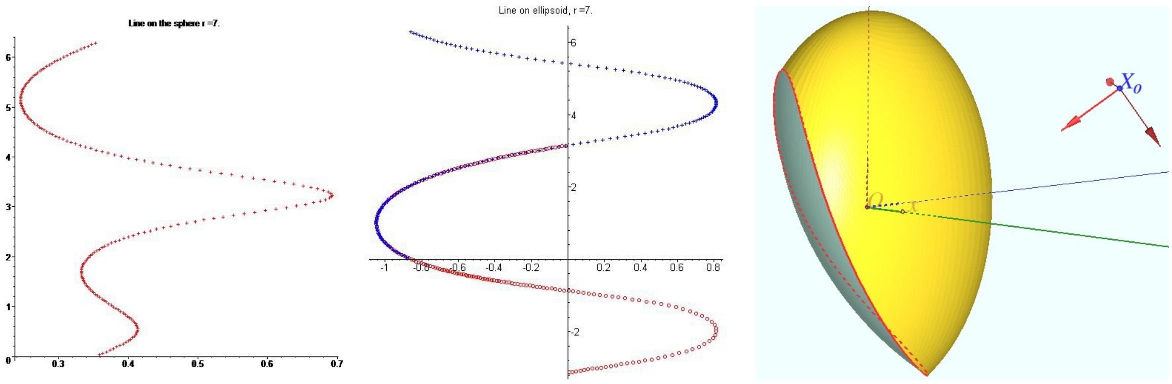

Figure 9: Intersection curves on the sphere and ellipsoid and intersection solid for $r=7$

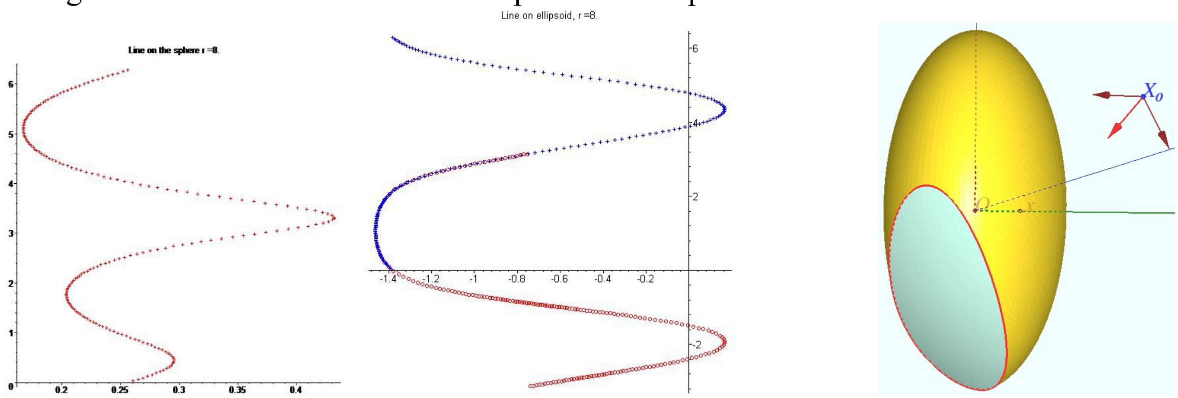

Figure 10: Intersection curves on the sphere and ellipsoid and intersection solid for $r=8$

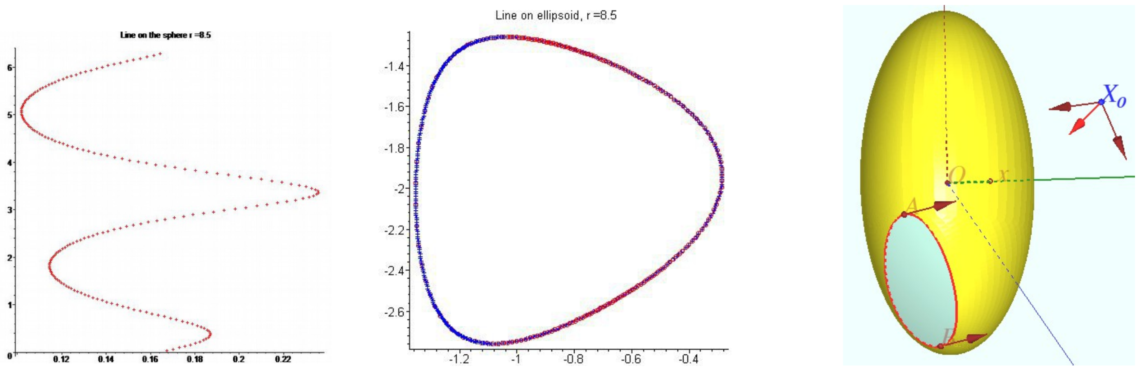

Figure 11: Intersection curves on the sphere and ellipsoid and intersection solid for $r=8.5$

We note two features of the GInMA images. Since the CAS of GInMA does not allow the solution of equations of degree to be higher than four, we need to perform two operations manually. In the second step of an active drawing in GInMA file. We need to put the points H and H' in positions to show perpendiculars to the surface of the ellipsoid passing through the point $\vec{X}_0$. The violet curve on the yellow ellipsoid shown on the left of Figure 12 is the locus of the point H and the blue curve is the locus for the point H'. Accordingly, the points H and H' allow the GInMA program to construct the spherical part inside the ellipsoid. At step 6 of an active drawing in GInMA file, we need to move the points A and B to the top and bottom of the intersection curve, respectively (see the right of Figure 12). These allow GInMA to fill the surface of the ellipsoid for cases when the vertices $(0,0,c)$ and $(0,0,-c)$ are not divided by the intersection curve. We note that if these points are not on the top and bottom as seen in the middle of Figure 12, GInMA possibly can show only part of the ellipsoidal surface.

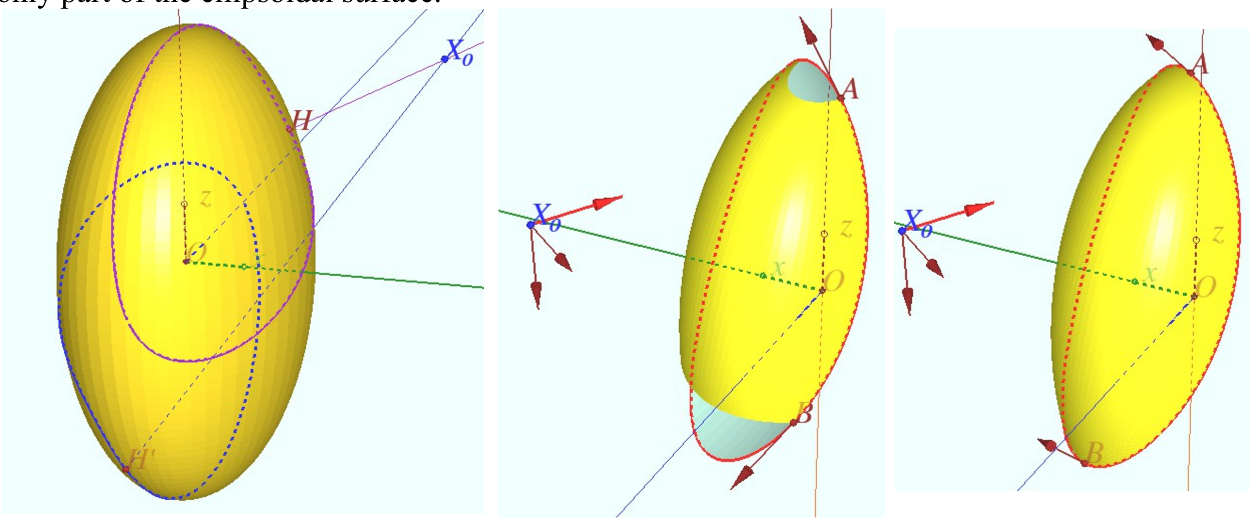

Figure 12: Active points in GInMA help to get correct solid of the intersection

6.6 The self-intersection point on the line of intersection

We consider the center of a sphere located at $\vec{X}_0^T=(3,4,2)$ and the ellipsoid with semi-axes of (2,3,7). This case will be referred to as Solid 1. For comparison purposes, we call the Example 3 in [2], when the center of a sphere is at $\vec{X}_0^T=(3.1, 3.35, 2.35)$ and the semi-axes of the ellipsoid are (2,3,7.4), as Solid 2.

To find the point of self-intersection for Solid 1, we solve the equation (2) for λ, and use it to find the radius of the sphere. It has four solutions. The smallest ($r_{min} \approx 2.50000$) and the largest ($r_{max} \approx 10.45295$) distance values define the range of the radius for which the intersection of the surfaces may exist. Radius $r \approx 7.519658$ defines the point of self-intersection C with the coordinates (−0.57963, −2.29261, 4.03336). Solution $r' \approx 7.55098$ determines the radius at which the second loop of the curve of intersection disappears. For the Solid 2 we obtain the smallest ($r_{min} \approx 2.19656$) and the largest ($r_{max} \approx 10.86524$) distance values, radius $r \approx 7.00206$, the point of self-intersection C (−0.689727, −2.323140, 3.925523) and $r' \approx 7.15438$. We refer to the point D on the right of Figure 12 and note that D is at the tangent of the intersecting curve and u=*constant*. At the point D, u has extremal value. The corresponding equation can be obtained from the condition $\frac{\partial f(u,v)}{\partial v}=0$ or

$$\begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ x-x_0 & y-y_0 & z-z_0 \\ x-x_0 & y-y_0 & z-z_0+r \end{vmatrix} = 0, \quad \text{or} \quad a^2(x-x_0)\,y = b^2 x\,(y-y_0).$$

The surface of the elliptical piece is calculated as in previous cases. The surface of the spherical piece is divided into two pieces within each of the loops. Since these two pieces have two respective acute angles at C (see right of Figure 13), we need to construct two separate coordinate systems for each loop. The z-axis we choose for each system should be close to the bisectors of the acute angle at C and the center of the loop.

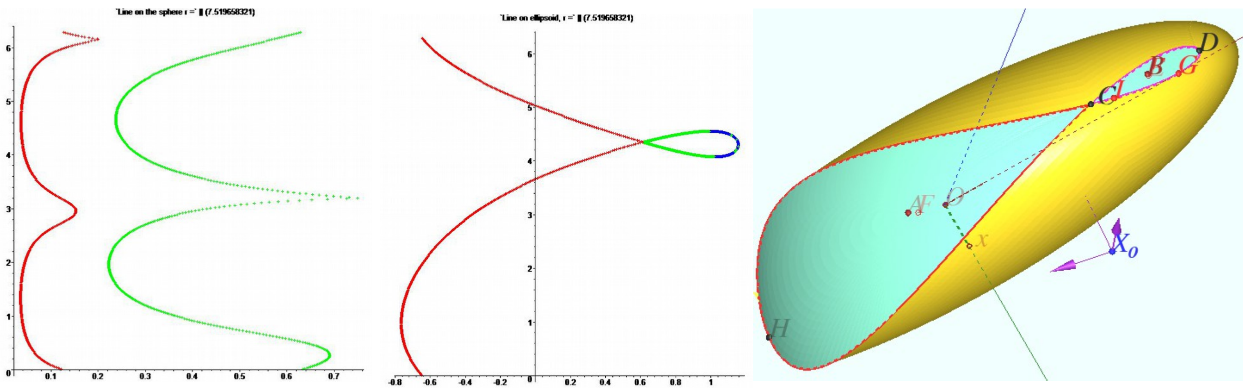

Figure 13: Intersection curves on the sphere and ellipsoid (2, 3, 7) and intersection solid.

We describe how we optimize the choices of A and B with the help of GInMA as follows:

1. We first select the points A and B in the interior of two separate intersecting loops respectively.

2. We drag the points A and B to achieve the best uniform shading of those two intersecting surface area of the sphere.

3. We use the direction X_0A as the z-axis when calculating the portion of the surface of the sphere with GInMA, which contains the intersecting loop with point A in it. Next, we set the point F to be on the ray of X_0A. Similarly, we use the direction X_0B as the z-axis for the second loop and use GInMA to calculate the portion of the surface of the sphere, which contains the intersecting loop with point B in it. Finally, we set the point E to be on the ray of X_0B.

4. We use these positions E and F in Maple for calculating the surface area for the sphere. We note the Maple file in [9], we use E as the *pointE* and F as the *pointF.*

The surface area of the ellipsoidal piece is divided into three parts. Two parts are inside the intersection curve loops and the third is the remaining portion of the ellipsoidal surface. We define the area within the loops. The area of the intersection solid is obtained by subtracting the area within the loops from the surface area of the ellipsoid. We find the intersection curve on the ellipsoid by solving the set of initial equations under the condition $u > u_c$ for one loop and $u < u_c$ for the other loop, where u_c is the coordinate of point C.

Figures 13 and 14 show the appearance of the solid in space, the curve of intersection in the coordinates of the sphere and in the coordinates of the ellipsoid. Accuracy of the calculations is determined by the convergence of the solution when the number of calculation points is varied. Accuracy of the calculations is also determined by comparison with the results of [2] for Solid 2.

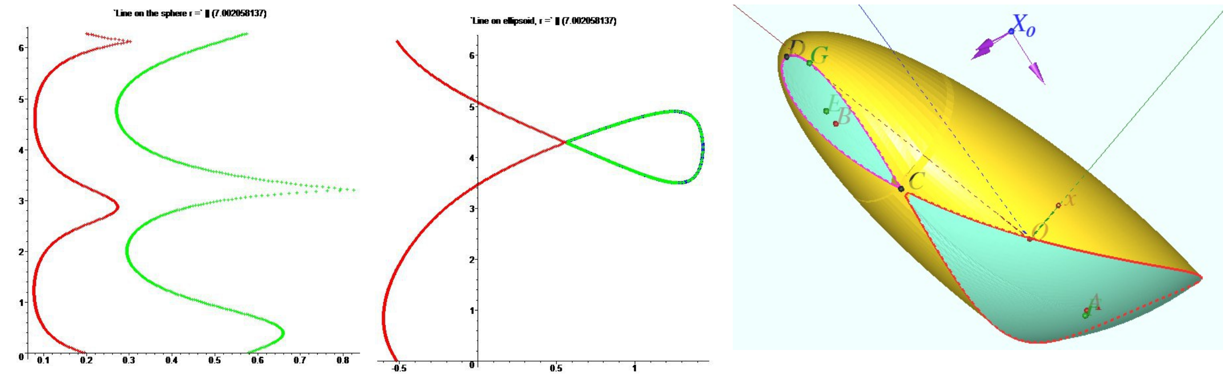

Figure 14: Intersection curves on the sphere and ellipsoid (2, 3, 7.4) and solid of intersection.

The set of calculations for Solid 1 and Solid 2 are shown in Table 5. We vary the number of the calculation points and show the results in table columns, where the area of the ellipsoid is shown in the upper value and the area of the sphere is shown in the lower value in each cell. We use equation (22) to obtain the values for infinity in the last table column.

n	100	300	550	∞
Solid 1: $S(2,3,7)$, $\vec{X}_0^T=(3,4,2)$ and $r \approx 7.519658$.	125.7601 28.8371	125.5526 28.8109	125.5003 28.8149	125.48 28.816
Solid 2: $S(2,3,7.4)$, $\vec{X}_0^T=(3.1, 3.35, 2.35)$ and $r \approx 7.00206$.	119.5152 32.2572	119.2160 32.27038	119.1462 32.27401	119.123 32.2743

Table 5

These values are in close agreement with those given in [2, Example 3]. The surface area of the relevant portions for the ellipsoid and the sphere are respectively 118.88 and 32.44.

7. Methods of finding solutions

Most of the methods we describe in this paper apply numerical integration schemes based on Maple in conjunction with the powerful visualization program GInMA. A problem that arises lies in the fact that for complicated geometrical problems it is typical that relevant equations do not admit a unique solution. We can solve quite complicated equations with Maple but usually there is no assurance that we are using the correct root. In such case, we use GInMA visualization capability to eliminate such extraneous root(s). It is important that the same computational grid, which is used to calculate the area, is being used to paint the surface. An example of such a situation is shown in Figure 10 on the right, the intersection solid image when $r=8$. Three features of the solution can be seen in this case. Firstly, the half of the ellipsoid surface lying opposite the boundary curve is lost. Secondly, an issue arises that requires a formal decision. It arises in conjunction with covering the part of the ellipsoid located outside the intersection solid. Thirdly, the reference points on the surface near the intersection curve above H' have a big step. This is clear by the form of a blue sphere which “peeps” through the holes in the yellow surface of the ellipsoid and by the "ribbing" of this part of the ellipsoid surface. During the calculation of the area such ribbing corresponds to a rapid decrease in calculation accuracy when we reduce the number of calculation points.

The equations in Maple and GInMA programs are written the same way and transferred from one program to another using “copy-paste” so the developer can be sure of the correctness of the calculation. Additionally, in deriving the equations of the intersection curve, the visual observation of the curve allows the developer to feel confident because the slightest breach is immediately visually observed (especially since the solution will be observed to jump from one branch to another). If the response can be observed, it is usually easier to find a "cure." During the creation of the Maple worksheets for this paper, step by step calculations and their GInMA visualizations have been performed. The worksheets demonstrate high performance and the ability to calculate the location of any variant of the sphere relative to the ellipsoid (assuming that its center lies outside the ellipsoid). This illustrates the power of the approach used.

8. Conclusions

A numerical procedure to quickly calculate the surface area of the intersection solid of a sphere and an ellipsoid has been discussed. Achieved accuracy of the calculation is close to $\delta(n)n^2 \sim 1$ and $\delta(n) \sim 10^{-5}$ for 300 calculation points. The calculation time for one solid usually is less than a minute on a typical university computer. A method was proposed for the numerical integration based on distributing calculation points obtained using the cosine law. It was theoretically justified and numerically confirmed that the calculation accuracy is of order n^{-2}. It was demonstrated that the numerical results agree with our earlier results given in [2].

The ability to visualize the results of calculations using a curve or a surface greatly assists in the choice of when the defining equations have multiple solutions. This situation is typical for the numerical computation of problems associated with complicated spatial 3D constructions. Joint use of a CAS such as Maple and a powerful visualization program such as GInMA can significantly decrease the time for the development of the solution.

The first step of the calculation is that of finding a range of radii for which the curve of intersection exists. This requires the solution of an equation of sixth degree and finding two of its real roots corresponding to the internal and external osculation of the ellipsoid and a sphere with variable radius and fixed center. Since GInMA currently allows only the solution of the fourth degree equation, an element of manual management is integrated in the solution. When possible, user should manually adjust the points so that the perpendiculars constructed from these points intersect with the center of the sphere. Thus these points become the bases of the perpendiculars dropped from the sphere center to the ellipsoid. These points can then be used to construct the solid of intersection.
The accuracy of the calculation becomes significantly better if the vertices $(0, 0, c)$ and $(0, 0, -c)$ are divided by the intersection curve. If the radius of the sphere is greater than the distance to some vertex but smaller than the distances to the vertices and it is advisable to rename the axes and to choose for the axis the one which passes through the vertex closest to the sphere's center. Similarly, if the sphere radius is smaller than the distance to some vertex but greater than the distances to the opposite vertices, it is advisable to rename the axes using for the z axis the one passing through the vertex farthest from the sphere's center. It transforms case shown in Figure 1 on the right in the case shown in Figure 1 on the left, transforms the range of $v \in [v_{min}, v_{max}]$ to the range of $v \in [0, 2\pi)$. Finally, we note that if the surface area of the intersection solid is substantially greater than the half of the ellipsoid (or the sphere), it is advisable to find the surface area of the remainder of the ellipsoid, find the total ellipsoid area and subtract remainder from the total ellipsoid area.

9. References

[1] G.A.Korn, and T.M.Korn. Mathematical Handbook for Scientists and Engineers. Definitions, Theorems, and Formulas for Reference and Review. DOVER PUBLICATIONS, INC. Mineola, New York.

[2] S. Thompson, W.–C. Yang, and V. Shelomovskii. *The Intersection of a Sphere and an Ellipsoid, A First Approach,* Electronic Journal of Mathematics and Technology (eJMT), Issue 1, Vol.7, 2013.

[3] W.–C. Yang, and V. Shelomovskii, *Mean Value Theorems in Higher Dimensions and Their Applications*, Electronic Journal of Mathematics and Technology (eJMT), Issue 1, Vol.6, 2012.

Software Packages

[4] [Maple] A product of Maplesoft, http://www.maplesoft.com/.

[5] [GInMA] GInMA, 2012, S. Nosulya, D. Shelomovskii, and V. Shelomovskii. http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx

Supplemental Electronic Materials

All GInMA supplemental materials that accompany this paper can be used from figures.
Install the GInMA software from the website and click on the picture.

[6] V. Shelomovskii, Maple worksheet for section 5.1.
[7] V. Shelomovskii, Maple worksheet for section 5.2.
[8] V. Shelomovskii, Maple worksheet for section 6.5.
[9] V. Shelomovskii, Maple worksheet for section 6.6.